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Laminar separation in buoyant channel flows

By VIJAY MODI† AND F. K. MOORE
Sibley School of Mechanical and Aerospace Engineering, Cornell University,
Ithaca, NY 14853, USA

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A slow moving flow in a duct emerging into a quiescent negatively buoyant environment may separate from its inner wall prior to the lip. Buoyancy accelerates the flow, curving the streamlines within the duct away from the walls. The resulting deceleration at the wall may be sufficient to provoke separation. The problem of the location of this separation point in a two-dimensional channel is studied. A potential-flow model is examined first to explore the large-Reynolds-number behaviour. The form of the potential-flow description in the vicinity of the assumed location of separation is characterized by the presence of a square-root singularity in the pressure gradient at the wall. This permits use of the ideas of viscous–inviscid interaction, proposed by Sychev (1972), to determine the separation location as a function of Froude and Reynolds numbers. Results obtained in the high-Reynolds-number limit show that the channel flow separates at shorter distances from the entrance as Froude number is reduced.

1. Introduction

Most buoyant jets that we observe, e.g. smoke stacks, cooling towers, faucets and orifice discharges, appear to 'separate' from the wall at the very lip of the duct they are leaving. Indeed more often than not they do so, but under certain conditions (usually low inertia-to-buoyancy ratio) they may 'separate' from the wall before they reach the lip of the duct. This separation prior to reaching the lip is usually accompanied by penetration of the negatively buoyant fluid into the duct. This phenomenon can impair the performance of a device like a cooling tower by reducing the available draught height. Jörg & Scorer (1967) performed an experimental study to determine the depth of this penetration, for laminar and turbulent flow in a duct of circular cross-section. They found that the presence and extent of penetration depends on many parameters: Reynolds number, Froude number, duct height-to-diameter ratio, upstream velocity profile, wall roughness and heat transfer through the duct wall. The Froude number (ratio of inertia to buoyant forces) emerges as the critical parameter, and penetration is found to increase with lower Froude numbers. However, they were unable either to eliminate or quantify the effect of the other parameters.

A clearer understanding of this problem may follow from theoretical study of an idealized steady, two-dimensional laminar flow undergoing separation, subject to buoyant, viscous and inertia forces in a model problem. In this paper the effects of those forces on the location of separation is studied in the limit of large Reynolds number (based on duct width).

† Present address: Department of Aeronautics and Astronautics, Gas Turbine Laboratory, Massachusetts Institute of Technology, Cambridge, MA 02139, USA.

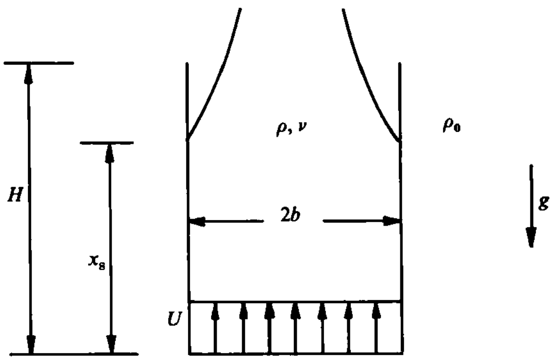

FIGURE 1. Schematic of the model problem considered.

The problem shown in figure 1 is considered. Fluid of density ρ and kinematic viscosity ν flows through a two-dimensional insulated duct of width $2b$ and height H. The flow at the upstream location is assumed to have a uniform velocity profile. At some downstream distance x_s the fluid breaks away from the wall to form a free jet in an otherwise stagnant environment of higher density ρ_0. Gravity g acts to provide a net body force per unit volume of $(\rho_0-\rho)g$ in the direction of motion. The resulting increase in the velocity forces the free jet to converge toward the duct centreline. Of course, the initially straight streamlines (far upstream in the duct) must begin to deflect inward even before the emergence of the jet, and this causes a deceleration near the wall. As a result, the wall boundary layer must presumably separate from the wall at the distance x_s from the duct entrance. It is our task to find x_s from the interaction of the flow phenomena just described.

The flow is assumed to be two-dimensional, incompressible, laminar and steady throughout. Moreover, surface tension is neglected. We define dimensionless variables by choosing b as the unit length and U as the unit velocity. Thus the problem is to find

$$x_s = x_s(Re, Fr) \quad \text{as } Re \rightarrow \infty, \tag{1}$$

where Reynolds number $Re = Ub/\nu$ and Froude number $Fr = \rho U^2/(\rho_0-\rho)gb$.

For large Reynolds numbers, the viscous forces will in fact become negligible in most of the flow field. Only in the boundary layers will viscosity remain important, and if the boundary layer separates from the wall, it will do so at a unique position. Beyond that position, the boundary layer will continue as a free vortex sheet, unable to support further streamwise pressure variation. To find the location of separation, it is necessary to consider both inviscid potential and viscous-boundary-layer flows.

First, we shall formulate and solve the potential-flow problem, with the location at which the free vortex sheet leaves the wall remaining unknown for the time being. This location will be determined later, from the requirement that the potential-flow and the limiting viscous-flow pictures are mutually consistent, using ideas of viscous–inviscid interaction postulated by Sychev (1972).

2. Potential flow

The domain of the potential-flow problem is shown in figure 2. The free vortex sheet is assumed to leave the wall at the point S, also referred to here as the point of breakaway to distinguish it as arising out of purely inviscid considerations. It is convenient to choose the origin of the coordinates at S. The part of the channel wall

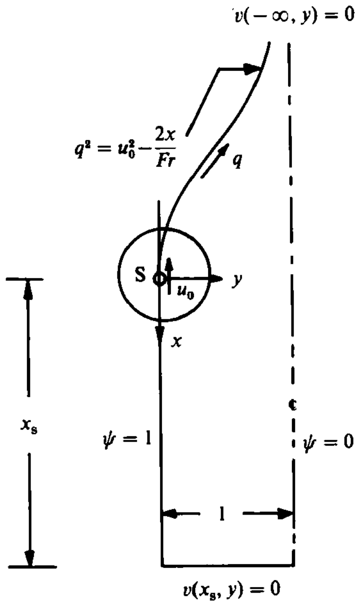

FIGURE 2. Schematic of the potential-flow problem.

beyond S (for which $x < 0$) is of no importance, since it is located in the stagnant separated region. Assuming symmetry, only half the problem shown in figure 1 need be considered. The wall and the symmetry boundary are streamlines. The stream function on the symmetry boundary ($y = 1$) is taken to be zero. It follows from the choice of dimensionless variables that $\psi = 1$ on the wall ($y = 0$) as well as the free surface. A second boundary condition on the free surface is that the static pressure equals that in the stagnant environment at that level. By use of the Bernoulli equation, this condition may be transformed into a statement about the flow speed on the free surface:

$$q^2 = u_0^2 - \frac{2x}{Fr}, \tag{2}$$

where u_0 is the flow speed at S. The velocity on the wall, and therefore u_0, are features of the solution to be found.

The foregoing boundary condition on the free surface (above S) requires that the square of the flow speed increases along the free surface, while on the same streamline below S, velocity normal to the wall is zero. This sudden change suggests the possibility of a singularity at S. To investigate this possibility we seek a description of the flow in the close vicinity of S. The governing equation is the Laplace equation, and the relevant boundary conditions are that the wall and the free surface constitute a single streamline, and the pressure condition (2) is satisfied at the free surface. Based on known potential-flow descriptions of how free streamlines leave a solid wall (see Thwaites 1960) we anticipate the following expansion for the complex velocity $u - iv$ in the vicinity of S:

$$u - iv = -(u_0 + kz^{\frac{1}{2}} + k_2 z + k_3 z^{\frac{3}{2}} + \ldots), \tag{3}$$

where $|z| = |x + iy|$ is small. In order for the wall upstream of S ($z = x > 0$) to be a streamline, each coefficient in the expansion (3) must be real:

$$\mathrm{Im}\,(k, k_2, k_3, \ldots) = 0. \tag{4}$$

We next specify that u_0 is not zero, that is the point S is not a stagnation point. Later we shall consider that possibility and conclude that it is unrealistic. With u_0 not zero, the free-surface shape in the vicinity of S is given by

$$y = \frac{2k}{3}\frac{(-x)^{\frac{3}{2}}}{u_0} + O(-x)^{\frac{5}{2}} \quad (x < 0). \tag{5}$$

Once the local shape of the free surface is known, the u- and v-velocities on the free surface can be found from (3). The requirement that these velocities satisfy the pressure boundary condition (2), leads to an additional constraint on the coefficients of the series, namely

$$\tfrac{5}{3}k^2 + 2u_0 k_2 = \frac{2}{Fr}. \tag{6}$$

The series in (3) together with constraints given by (4) and (6) comprise a potential-flow solution in the vicinity of S, correct to $O(x)$. The actual numerical values of u_0 and k will depend on the Froude number; however, this dependence can only be found from the solution of the potential problem over the full domain. The higher-order coefficient k_2 can then be determined from (6) if necessary. In principle, the pressure boundary condition (2) can be satisfied to any desired order providing additional constraints similar in spirit to (6). We shall, however, only be interested in the leading-order quantities u_0 and k.

The above analysis also shows that the potential-flow solution in the vicinity of S for this problem has the same form (given by (3)) as that of the known solutions for free streamlines leaving a solid wall. The particular geometry of the problem and buoyancy affect the local description inasmuch as they determine the numerical values of the coefficients in (3). Equation (5) shows that the free surface leaves the wall tangentially but with infinite curvature at S. Thus, if the wall itself has finite curvature, the leading coefficient k must be positive in order that the free surface does not enter the wall.

From (3) the velocity on the wall in the upstream vicinity of S can be written as

$$u = -u_0 - kx^{\frac{1}{2}} + O(x) \quad (x > 0). \tag{7}$$

From Bernoulli's Law, one can then infer the pressure along the wall in the vicinity of S, with reference to some pressure p_0 (say) and density ρ, to be

$$p = p_0 - \rho u_0 k x^{\frac{1}{2}} + O(x) \quad (x > 0).$$

For non-vanishing k, the square-root term in the pressure leads to a singular pressure gradient at S, given by

$$p_x = \frac{-\rho u_0 k}{2x^{\frac{1}{2}}} + O(1) \quad (x > 0). \tag{8}$$

This square-root singularity is characteristic of known solutions of the theory of ideal fluids with free streamlines. (See for example, Thwaites 1960.)

The pressure gradient implied by this inviscid solution in the vicinity of breakaway will be important when we study the effect of non-zero viscosity. Therefore, we particularly need to know both the strength of the singularity k and the velocity u_0 as functions of the problem parameters, that is,

$$k = k(Fr, x_S), \quad u_0 = u_0(Fr, x_S).$$

To determine the above relationships, we return to the full potential-flow problem shown in figure 2.

Because the fluid velocity is not uniform in magnitude on the free surface, classical free-streamline theory cannot be used to solve this problem analytically. The principal difficulty arises from the fact that unless the magnitude of the velocity is constant on the free streamline, one cannot determine the domain in the hodograph plane beforehand. Hence we shall seek a numerical solution to the problem. In addition to the boundary conditions on the wall and the free surface, we shall also need suitable boundary conditions at the upstream, downstream and symmetry boundaries. At the symmetry boundary ($y = 1$), the stream function $\psi = 0$ as noted earlier. At the upstream boundary ($x = x_s$) we specify $v(x_s, y) = 0$. This is an artificial boundary condition since only in the limit of $x_s \rightarrow \infty$ would it imply $-u(x_s, y) = 1$. The downstream boundary condition is $v(-\infty, y) = 0$. Since the jet thickness approaches zero as $x \rightarrow -\infty$, this boundary condition is appropriate. Actual numerical calculations, however, will be carried out with truncation of the boundary at a finite distance.

Since the shape of the free surface is unknown *a priori*, an iteration procedure will be needed to find a potential-flow solution. The simplest procedure would be first to solve the linear problem of two-dimensional potential flow in a reasonably guessed domain. The resulting flow speed on the free surface would presumably not satisfy the boundary condition (2). One would then suitably update the domain boundary to better satisfy that boundary condition. One would repeat these two steps until the free surface boundary condition was well satisfied.

For reasons cited below, the Boundary-Integral-Equation Method (BIEM) seems particularly attractive for the first step. The numerical computations are performed only on the boundary of the domain being considered, so that the effective dimensionality of the problem is reduced by one. Moreover, interior solutions can be obtained subsequently by a suitable marching procedure if needed. But this is not necessary at every iteration because the surface velocity needed to update the domain as well as the strength of the singularity are sole functions of boundary quantities. The BIEM uses shape functions to represent functional values between discrete nodes. If the nature of the singularity is known, it can be used to substitute a special interpolation function (near the singularity) which more closely approximates the exact result (Liggett & Liu 1983). On the free surface, in the downstream vicinity of S, where $-x < \epsilon$, a linear interpolation function to represent the flow-speed behaviour is consistent with (2), provided $u_0 \gg (2\epsilon/Fr)^{\frac{1}{2}}$. Whereas on the wall, in the upstream vicinity of S, a square-root-type interpolation function is consistent with (7). Our calculations, however, are carried out with a linear interpolation function over the entire boundary. To overcome this shortcoming, calculations are repeated with progressively smaller nodal distances in the vicinity of S, until a converged solution is found.

The foregoing procedure allows us to find accurate solutions for $Fr > 0.5$. If the Froude number is reduced below 0.5, it seems possible that u_0 may vanish. In such a situation the description of the flow in the vicinity of S given by (6)–(8) is no longer valid, and with u_0 zero, a linear interpolation function near S would no longer be suitable. A more complete discussion of this problem and the details of the BIEM numerical procedure are given by Modi (1984). Since our subsequent analysis is valid for large Froude numbers only, the inability to find accurate potential-flow solutions for $Fr < 0.5$ does not pose a serious limitation, and we may use the BIEM to provide the shape of the free surface as well as the velocity on the wall.

The results for the wall velocity ($-u$) are shown in figure 3 for four different values

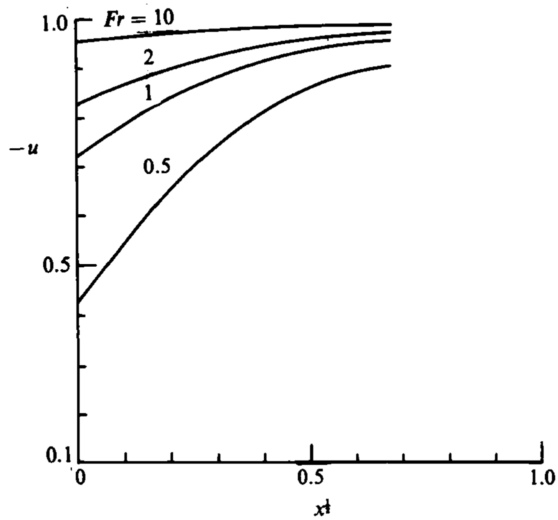

FIGURE 3. Wall velocity $-u$ for various Froude numbers.

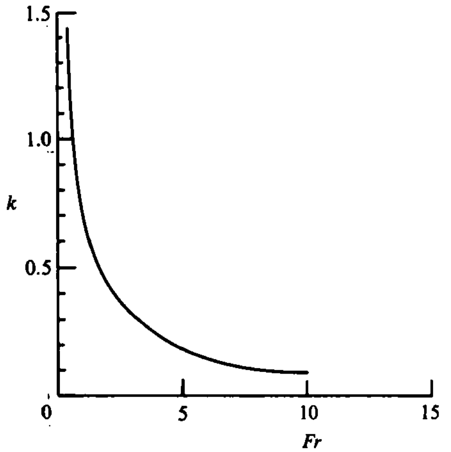

FIGURE 4. Strength of singularity k as a function of Froude number.

of Fr, with x_s fixed at 0.5. The intercept and the slope of these curves at $x = 0$ provide us with u_0 and k for each Fr according to (7). Numerical calculations with other values of x_s reveal that the wall velocity profile near S (and thus u_0 and k) is insensitive to the actual value of x_s provided x_s is larger than 0.5. The downstream boundary condition was located far enough from S to ensure insignificant influence on u_0 and k as well. Thus, our numerical solution does in fact represent that for an infinite jet and a semi-infinite cylindrical duct, for which the solution is obviously independent of the value of x_s. The artificial boundary conditions prescribed at the start and end of the computational domain thus prove to be effective.

Both k and u_0 are obtained as functions of Fr alone. The variation of k with Fr is shown in figure 4 and that of u_0 with Fr in figure 5. We observe that the strength of the singularity k diminishes with increasing Fr and it should eventually go to zero as Fr goes to infinity. The velocity u_0 at S increases with Fr and should eventually

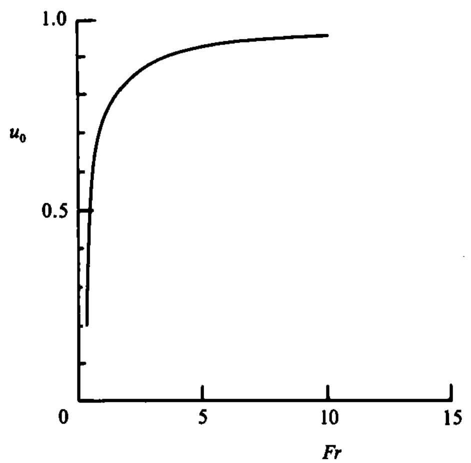

FIGURE 5. Flow speed at breakaway u_0 as a function of Froude number.

go to unity as Fr goes to infinity. For large Fr, the variation of k and u_0 with Fr is well represented by

$$k \approx \frac{1}{Fr}; \quad u_0 \approx 1 - \frac{0.4}{Fr}, \quad (Fr \gg 1). \tag{9}$$

The same potential-flow problem was solved independently by Vanden-Broeck (1984) using a collocation scheme. He presents results for u_0 and the shape of the free surface as a function of the Froude number. Inferring k from his results would require additional calculation; however our u_0 values are in good agreement with those obtained by him. This comparison is made for $Fr > 0.5$ only, since our numerical procedure does not allow us to find accurate solutions at lower Froude numbers. With the inviscid flow in the vicinity of breakaway (S) known, we may now address the question of determining x_s itself, as a matter of boundary-layer separation.

3. Viscous–inviscid interaction

In order to fix the correct physical location of the separation point, we must study the viscous flow inside the boundary-layer region. It is expected that this will remove the indeterminacy in the location of S. Indeed, on physical grounds, we expect the solution to be uniquely defined in terms of the two non-dimensional numbers Re and Fr, as we already have indicated in (1).

Consider first the situation in the classical boundary-layer approach. To study the applicability of any of our potential-flow solutions, we might use the wall pressure gradient (8) found previously to integrate the boundary-layer equations. The failure of such an approach is now well known (see Smith 1982 for a review). The assumed potential flow, with separation in the duct at a finite position x_s and a non-zero Froude number, is physically not possible in the limit of infinite Reynolds number. In fact, if the Froude number remains finite while the Reynolds number approaches infinity, the only possibility for a physical solution is that the separation occurs right at the entrance of the cylindrical duct, at the position satisfying the Brillouin–Villat condition (Birkhoff & Zarantonello 1957).

In order for the separation to occur inside the cylindrical duct itself, which is the case of present interest, the only possibility is that the Froude number is also high when the Reynolds number is high. A high Froude number implies a weak pressure singularity ($k \to 0$ when $Fr \to \infty$ in (8)). In the presence of a weak pressure singularity, Sychev (1972) proposed a consistent description of separation. The theoretical picture proposed by Sychev was a generalization of the 'triple-deck' flow structure discovered earlier by Stewartson & Williams (1969) and by Messiter (1970). Sychev's theory of viscous–inviscid interaction at separation is based on the same outer flow as described by (7) and (8). He finds that the main flow in the boundary layer is inviscid, owing to the large pressure gradient near separation. Consequently, viscous forces are important only in a thin sublayer adjacent to the wall. The flow acquires a three-region character, comprising the outer potential flow, the inviscid main boundary layer and the viscous sublayer. Well upstream of the interaction region, the pressure gradient will be negligible as a result of the assumption of large Froude number and parallel-walled geometry. It is therefore evident that the wall shear at this location will be described by classical boundary layer theory as

$$u_y \approx Re^{\frac{1}{2}}\tau,$$

where τ for a parallel-walled channel might be approximated by the Blasius semi-infinite flat-plate solution as

$$\tau = \frac{0.332}{x^{\frac{1}{2}}}. \tag{10}$$

On examination of the flow on a very small $O(Re^{-\frac{3}{8}})$ lengthscale, Sychev postulated that outer and boundary-layer flows would be possible in the interaction region. The problem depends upon the parameters k, u_0, ρ, ν and τ and requires that $Re^{\frac{1}{16}}$ be large. To verify Sychev's hypothesis, Smith (1977) sought a numerical solution. By suitable normalizations he first reduced the problem to one involving a single parameter, K, given by

$$K = \frac{Re^{\frac{1}{16}} u_0^{\frac{3}{4}}}{(\tau)^{\frac{9}{8}}} k. \tag{11}$$

For a value of $K = 0.44$, his results did indeed suggest the existence of a meaningful solution. He could not find a converged solution for other values of K. Smith's calculations were the best evidence for Sychev's theory, in the absence of a mathematical proof of existence and uniqueness of the solution. Since then Korolev (1980) as well as Van Dommelen & Shen (1984) have used more accurate schemes and confirmed Smith's findings. With numerical investigations pointing quite firmly to both the existence and uniqueness of a solution, there is significant confidence in Sychev's theory. On this basis, we shall proceed to determine the location of the separation point; that is, the value of x_s.

With $K = 0.44$, and $k(Fr)$ and $u_0(Fr)$ from figures 4 and 5, we can find $x_s(Re, Fr)$ from (10) and (11). For $Re = 500$ and 5000, the calculated separation location is shown in figure 6 as a function of Froude number. Since we know the behaviour of $k(Fr)$ and $u_0(Fr)$ at large Froude numbers from (10), we can find an explicit expression for x_s valid at large Fr,

$$x_s = 0.025\left[\frac{Fr}{Re^{\frac{1}{16}}}\right]^{\frac{16}{9}}, \quad \text{for } Re^{\frac{1}{16}} \gg 1, \quad Fr \gg 1. \tag{12}$$

This expression represents the straight portion of the theoretical curves in figure 6.

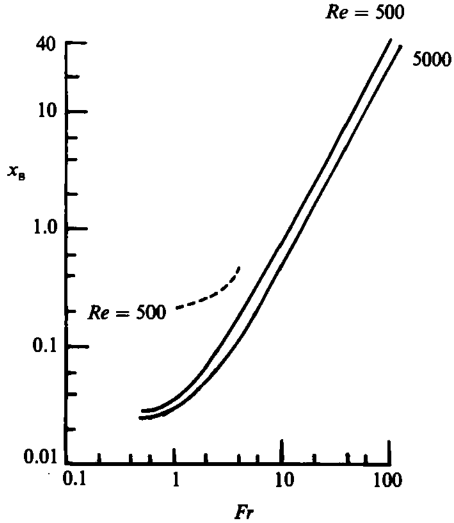

FIGURE 6. Separation location x_s (distance from the duct entrance) as a function of Fr calculated for $Re = 500$ and 5000 (——), based on Sychev's theory. A Navier–Stokes calculation for $Re = 500$ is also shown (----).

We also carried out a Navier–Stokes calculation (see Modi 1984) of the same problem for comparison. The calculation was carried out for a duct of dimensionless height 0.5 at $Re = 500$. The variation of the location of separation in the duct with Froude number is also shown in figure 6. For Froude numbers larger than about 5, the duct separates at its lip ($x_s = 0.5$) and thus only the results for lower Froude numbers are shown. It would be desirable to study longer ducts (so that separation occurs at larger Froude numbers) and larger Reynolds numbers, but both are computationally difficult. In spite of this limitation, it is encouraging to see that the prediction based on Sychev's theory follows a trend consistent with the numerical calculation.

Before continuing, it should be pointed out that the triple-deck value of $K = 0.44$ used above was obtained for a single fluid, whereas the case considered here is that of two fluids with different densities. In using the value for a single fluid here, we have assumed that the asymptotic solution near separation is not affected by the presence of the gravity force. For this assumption to be valid, the gravity forces must be small compared to the pressure forces in the viscous region. Using the triple-deck scalings for pressure and length, we find that the ratio $\Delta\rho g/p_x$ of the neglected gravity forces to the pressure forces is $O(Re^{-\frac{1}{8}}Fr^{-1})$. The estimate indicates that this ratio is small in the large $Re^{\frac{1}{16}}$, Fr limit considered here, justifying our use of $K = 0.44$ obtained for a single fluid. The gravity forces will of course alter the behaviour of the flow downstream of separation in the separated flow.

Note that, for the parallel-walled geometry under consideration, the strength of the singularity reduces with increasing Froude number but never becomes zero or negative. Thus the inviscid free streamline can break away from the wall at any arbitrary location. In this regard, the situation here is quite different from that arising in the study of separation over airfoils or bluff bodies (see Cheng & Smith 1982; Smith 1979) where k vanishes at some location on the body depending on the shape of the body. Consequently, in those studies the breakaway position to leading order depends upon the shape of body through the 'smooth separation' requirement.

As noted earlier, for Froude numbers less than 0.5, the possibility of vanishing u_0

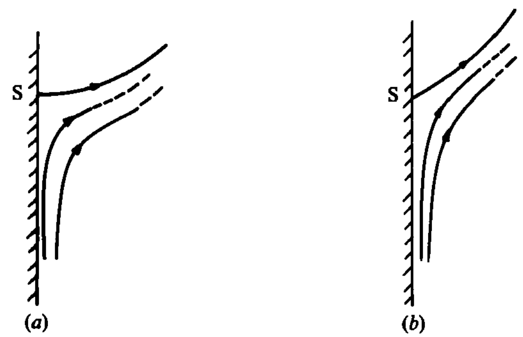

FIGURE 7. Schematic of two possible flow configurations at S if $u_0 = 0$. (a) Leaving angle $\frac{1}{2}\pi$ with $k = 0$; (b) leaving angle $\frac{2}{3}\pi$ with $k \neq 0$.

cannot be ruled out, and in such a circumstance the description of the flow in the vicinity of S is no longer given by (6)–(8).

For vanishing u_0, examination of (5) shows that there are two distinct potential-flow descriptions possible in the vicinity of S, depending on whether k vanishes or not. With $k = 0$ and $k_2 \neq 0$, the free surface at S makes an included angle of $\frac{1}{2}\pi$ with the wall. With $k \neq 0$, the free surface at S makes an included angle of $\frac{2}{3}\pi$ with the wall. The flow configurations corresponding to these solutions are sketched in figure 7(a, b). The boundary layers associated with these stagnation-point flows would experience an adverse pressure gradient over a finite distance in the limit of large Reynolds number. Consequently, non-interactive separation in the classical boundary-layer-theory sense would be expected ahead of any assumed location of separation S. If so, the only possibility for a physical solution would be separation ocurring right at the entrance of the cylindrical duct. The results shown in figure 6 do not apply to separation occurring in such a fashion, and we believe that any possibility for stagnation where the flow leaves the wall would belong to flow regimes, perhaps unsteady, which are fundamentally different from that considered here.

4. Discussion

In this paper we have found the separation location $x_s(Fr, Re)$ for a buoyant flow in a two-dimensional insulated duct in the limit of large Reynolds and Froude numbers. The solution to the interaction problem depended on the history of the boundary layer through the normalized shear τ. The assumptions of cylindrical duct shape, constant velocity profile at duct entrance and laminar flow allowed us to estimate τ from the Blasius flat-plate solution (10), where the duct is also the proper development length for the laminar boundary layer. We may briefly note the effects of relaxing each of those assumptions.

If the duct is convergent rather than cylindrical, the variation of τ with distance along the wall is no longer given by (10), but could be estimated from conventional boundary-layer theory. We would also need to evaluate u_0 and k for the new geometry; they would now be functions of x_s as well as Fr. In view of this fact, it would not be possible to write an explicit expression like (12) for x_s. Rather, it would be necessary to solve (11) iteratively to find $x_s(Fr, Re)$. The strength of the singularity k for a convergent duct would be significantly lower than that of a cylindrical duct at a fixed Fr, because a convergent shape would tend to oppose deceleration of the flow along the wall. A lower k and a larger τ compared to those of a cylindrical duct would permit the flow in a convergent duct to remain attached to the wall to larger distances.

The flow entering a duct might have a finite boundary-layer thickness, in contrast to the Blasius boundary layer discussed earlier. This also would contribute to an altered τ.

Finally, we note that consideration of the high-Reynolds-number limit and the assumption of steady laminar flow prevent us from obtaining any description of the flow beyond the separation point, where some sort of mixing process must occur. Also, one must be aware that the density gradient across the inclined free surface may lead to flow instability. Future research should consider the unsteady, three-dimensional flow which might result.

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